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Quaternion Wave Equations in Curved Space-Time

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Abstract

The quaternion formulation of relativistic quantum theory is extended to include curvilinear coordinates and curved space-time. This provides a promising framework for further exploration of a unified quantum/gravity theory.

1. *Introduction*

One of the most dramatic problems remaining in fundamental physics is that of combining quantum and gravity theory in a relativistic, second quantized formalism. A necessary prerequisite, it seems to me, is to find a 'natural' mathematical structure of quantum theory and for gravitation and then begin to bridge the gap between them by extending the quantum theory into curved space-time (Brill & Wheeler, 1957). This paper attempts to demonstrate such a natural structure and to show, fairly explicitly, how to extend the quaternion quantum theory into curved space-time. Rastall (1964) discussed the general nature of this problem. Here we try to develop more specific formulas for the transformations, covariant derivatives, and space-time dependence of the quanternion basis elements.

2. *Fields in Curved Space-Time*

We have identified six basic quaternion fields. They are distinguished by their transformation properties under a change of coordinate reference frame and by their structure in a curved space. The simplest, of course, is the scalar field $\phi(x)$. Four others, which seem directly applicable to particle physics are (Edmonds, 1973a) the 4-vector $V \equiv V^\mu b_\mu$, the axial 4-vector $a \equiv a^\mu \mathrm{a}_\mu$, the *a*-

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spinor $\psi_a \equiv \psi_a{}^\mu k_\mu$, and the v-spinor $\psi_v \equiv \psi_v{}^\mu l_\mu$. Here, $\{b_\mu\}$, $\{a_\mu\}$, $\{k_\mu\}$, and $\{l_\mu\}$ are the quaternion basis elements, which have their own coordinate or curvature forms, which we shall discuss. In the flat space Cartesian reference frame these all can be reduced to $\{e_\mu\}$ (Edmonds, 1972). The sixth field is the space-time curvature field itself, $L \equiv L^\mu e_\mu$. It has four complex functions L^μ at each event x. Knowing $L(x)$ means knowing the space-time curvature in coordinates which reduce to Cartesian coordinates in the flat limit, i.e. $\{b_\mu\} \to \{e_\mu\}$ as the curvature goes to zero. We shall discuss other coordinate systems shortly. Usually, one says that $g_{\mu\nu}(x) = g_{\nu\mu}(x) = g_{\mu\nu}{}^*$ gives a ten function description of the coordinate system and the space-time curvature. In the quaternion formalism we are developing here, $g_{\mu\nu}$ is a function of the eight real functions in the quaternion L.

The 4-vector, basis vector, quaternions are defined to be:

$$b_\mu(x) \equiv L^* e_\mu L \equiv b_\mu(x)^{(\alpha)} e_\alpha \tag{2.1}$$

We see that $b_\mu = b_\mu{}^*$ since $e_\mu = e_\mu{}^*$. The four complex functions L^μ then generate the sixteen real functions $b_\mu{}^{(\alpha)}$. One way to characterize a truly curved space is to say that no global coordinate transformation exists from the given coordinate system to a Cartesian coordinate system. (For example, the surface of a sphere is a curved two-dimensional space which can be mapped into the tangent plane, invertably, except for the point on the sphere opposite the point of tangency.) In quaternion form this says that the inverse quaternion L^{-1}, where $LL^{-1} \equiv e_0$, does not exist for all x. Otherwise, we could calculate:

$$(L^*)^{-1} b_\mu L^{-1} = e_\mu \tag{2.2}$$

for all x, and hence transform to flat space Cartesian coordinates.

The metric tensor is then defined to be:

$$g_{\mu\nu}(x) \equiv \tfrac{1}{2}[(b_\mu{}^\ddagger b_\nu) + (\ )^\ddagger] \equiv (b_\mu | b_\nu) \tag{2.3}$$

To insure that $g_{\mu\nu}$ is real, in curved space, we should perhaps modify this to:

$$g_{\mu\nu} \equiv \tfrac{1}{4}[(b_\mu{}^\ddagger b_\nu + b_\nu b_\mu{}^\ddagger) + (\ )^\ddagger] \tag{2.4}$$

but this is not really necessary since b_μ is real. The usual flat space metric will also be needed. It is a special case of equation (2.3):

$$\eta_{\mu\nu} \equiv \tfrac{1}{2}[(e_\mu{}^\ddagger e_\nu) + (\ )^\ddagger] \leftrightarrow \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \tag{2.5}$$

From equation (2.1) and (2.4), we can express $g_{\mu\nu}$ directly in terms of $L^\mu e_\mu$.

We have previously written the curvative equation in terms of the 4-vector basis elements $\{b_\mu\}$ (Edmonds, 1973a),

$$b_\mu{}^\ddagger (D^\mu D^\nu - D^\nu D^\mu) b_\nu = \kappa b_\mu{}^\ddagger M^{\mu\nu} b_\nu \tag{2.6}$$

The non-commutativity of the covariant derivative components (acting on a 4-vector) is another way to express the curvatuve of a space. Here the curvature is nonzero when $M^{\mu\nu}$, the 'matter tensor' is nonzero at some event x. Using equation (2.1), we can convert this into a partial differential equation for $\{L^\mu\}$ instead of for $\{b_\mu\}$. I do not know if this would make actual applications easier to solve but certainly we must find the $\{L_\mu\}$ either from $\{b_\mu\}$, using equation (2.1), or directly from equation (2.6). In Edmonds, 1973a, we find explicit expressions for $D_\nu D_\mu b_\lambda$ in terms of $\Gamma_\mu{}^\lambda{}_\nu$ and $D_\mu b_\lambda$ and expressions for $\Gamma_\mu{}^\lambda{}_\nu$ and $D_\mu b_\lambda$ in terms of $\{\partial_\mu b_\lambda{}^{(\alpha)}, b_\lambda{}^{(\alpha)}\}$. Hence equation (2.6) is a well-defined nonlinear second-order partial differential equation for $\{b_\mu{}^{(\alpha)}\}$. The $\{\Gamma_\mu{}^\lambda{}_\nu\}$ is defined by the requirement $D^\mu g_{\nu\lambda} \equiv 0$. However, in curved space this does not mean that $D^\mu b_\nu(x) = 0$, except in the 'neighborhood' of a single chosen even x_0. As a result we have a new subtlety in curved space quantum theory.

The usual axiom for generating quantum equations is $P = P^\mu e_\mu \to i\hbar\partial = i\hbar\partial^\mu e_\mu$. In curved space-time this would become $i\hbar D^\mu b_\mu$ or $i\hbar b_\mu D^\mu$ and these two possibilities are not equivalent physically. Similarly $D = b_\mu D^\mu \Rightarrow D^\ddagger = D^\mu b_\mu{}^\ddagger$ or $b_\mu{}^\ddagger D_\mu$ which are not equivalent. This may mean interesting inequivalent 'subcatagories' of wave equations describing the fundamental particles, i.e. several kinds of Dirac equations and Maxwell equations. This point really needs to be looked into.

So far, we have only considered the curved space properties of $\{b_\mu\}$. We have seen that a curvature inducing 'wave' equation can be defined for $\{b_\mu\} \leftrightarrow \{L^\mu\}$. Consider then the other quaternion types $\{k_\mu, l_\mu, \mathrm{a}_\mu\}$. We must specify how they depend on the space curvature. We do this as follows:

$$k_\mu(x) \equiv L^\ddagger e_\mu, \qquad l_\mu(x) \equiv L^* e_\mu, \qquad \text{and} \qquad \mathrm{a}_\mu(x) \equiv L^\ddagger e_\mu L \tag{2.7}$$

At this point, these definitions may not seem well motivated but they are rather logical choices based on the usual flat space Lorentz transformation structure of each type (Edmonds, 1972). Often it is convenient to consider linear combinations of the basis quaternions in constructing wave functions. For example, we usually write $\psi_a = \Psi_a{}^\mu f_\mu$ where $f_\mu \equiv f_\mu{}^{(\alpha)} k_\alpha$, e.g.,

$$\{f_\mu\} \equiv \{(e_0 + e_3), (e_1 - ie_2), (e_1 + ie_2), (e_0 - e_3)\}.$$

This gives:

$$f_\mu \equiv f_\mu{}^{(\alpha)} k_\alpha = f_\mu{}^{(\alpha)} L^\ddagger e_\alpha = L^\ddagger (f_\mu{}^{(\alpha)} e_\alpha) \tag{2.8}$$

which describes the curved space-time dependence of the spin eigen-state basis vectors. A similar construction holds for $\{l_\mu\}$ and $\{\mathrm{a}_\mu\}$.

We must next consider the covariant derivates of these basis elements, in order to construct coupled wave equations. We have, for example:

$$D^\mu k_\nu = D^\mu(L^\ddagger e_\nu) = D^\mu(L^\lambda e_\lambda{}^\ddagger e_\nu) = D^\mu(L^\lambda) e_\lambda{}^\ddagger e_\nu \tag{2.9}$$

Thus, knowledge of $D^\mu(L^\lambda)$ from $D^\mu(b_\nu)$ and $b_\nu \equiv L^* e_\nu L$ gives us (implicitly at least) a complete description of the needed covariant derivatives. We see again that $\{L^\mu\}$ is more 'fundamental' in the basic structure than $\{b_\mu{}^{(\alpha)}\}$.

3. *Invariant Field Equations*

With the structure we have developed so far, we can consider the formation of partial differential field equations. We do this by postulating two very important basis principles (Edmonds, 1973a, Rastal, 1964).

Invariance Principle

All wave equations are formed out of *invariant* quaternion units. This insures that the equations are 'form invariant' for any invertable coordinate transformation (General Relativity).

Lorentz Principle

All wave equations must be form invariant under a special symmetry group which transforms each quaternion basis element:

$$b_\mu \to \mathscr{L}^* b_\mu \mathscr{L}, \qquad a_\mu \to \mathscr{L}^\ddagger a_\mu \mathscr{L}, \qquad k_\mu \to \mathscr{L}^\ddagger k_\mu, \qquad l_\mu \to \mathscr{L}^* l_\mu,$$
$$b^\mu \to \mathscr{L}^* b^\mu \mathscr{L},$$

where $\mathscr{L}\mathscr{L}^\ddagger \equiv 1$. In other words, only those wave equations which are unaltered by the above substitutions are considered applicable to the physical world. It is currently a mystery to me as to why God would put such a restriction on the design of the universe. (Perhaps it is related to the conservation laws.) This is a stringent restriction on nature and no doubt has some deep motivation at present unclear. I think that this requirement is equivalent to saying that all wave equations belong to some representation of the ordinary Lorentz group, even in curved space-time and accelerated reference frames. One used to say that Lorentz invariance made all inertial observers equivalent (no preferred frame). This is not true, however. The three-degree background radiation apparently singles out a preferred reference frame so that one can define absolute motion relative to the 'universe' or the 'vacuum'. The invariant quaternion structure makes all laws form invariant for any reference frame. This is not incompatible with a special boundary condition on the universe singling out a special frame (the big bang at $t = 0$). We see the Lorentz principle as a mathematical symmetry, really unconnected (except historically) with any particular reference frames.

We could, for example, write the following coupled equations.

$$(P - (\epsilon/c)A)\psi_a = mc\psi_v, \qquad P = i\hbar D, \qquad P^\mu = i\hbar D^\mu$$

$$(P^\ddagger - (\epsilon/c)A^\ddagger)\psi_v = mc\psi_a, \qquad D = b_\mu D^\mu, \qquad \psi \equiv \begin{pmatrix}\psi_a\\ \psi_v\end{pmatrix}$$

$$(P|P)A - P(P/A) = m_\gamma{}^2 c^2 A + \epsilon\left(\left[\psi^*\begin{pmatrix} b_\mu & 0\\ 0 & b_\mu{}^\ddagger\end{pmatrix}\psi\right] + [\ \ ]^\ddagger\right) b^\mu$$

$$b_\mu{}^\ddagger(D^\mu D^\nu - D^\nu D^\mu)b_\nu = \kappa b_\mu{}^\ddagger\left[\left\{\frac{1}{m}\psi^*\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}(D^\mu\psi)\psi^*\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}(D^\nu\psi)\right\}\right.$$
$$\left. + \{\ \ \}^\ddagger + (D^\mu A)(D^\nu A^\ddagger)\right] b_\nu \qquad (3.1)$$

These equations probably do not meet all the physical requirements of a reasonable unified theory but are intended to demonstrate the Invariance Principle and Lorentz Principle. It is easy to check that both are satisfied. The quaternion structure and these two restrictions do not leave very many choices as to equation structure.

The above equations are not well-defined until we specify how to calculate $D^\mu A^\nu, D^\mu \psi_a{}^\mu$, and $D^\mu \psi_\nu{}^\nu$. We obtain these by requiring that $D^\mu A = D^\mu(A^\mu b_\nu)$ transform like P^μ, since the quaternion A is invariant. We have $D^\mu(A^\nu b_\nu) = (D^\mu A^\nu) b_\nu + A^\nu(D^\mu b_\nu)$ and $D^\mu b_\nu$ is known. In this particular case we can proceed differently. Since $D^\mu(g_{\nu\lambda}) = 0$, we can define $D^\mu A^\nu$ to transform like $D^\mu(g^{\nu\lambda} b_\lambda)$. However, in $\psi_a{}^\mu k_\mu$ we expect that $\psi_a{}^\mu$ transforms not like $g^{\mu\nu} k_\nu$ but like $(k^\mu | k^\nu) k_\nu$, where $(k^\mu | k^\nu)(k_\nu | k_\lambda) \equiv \delta_\lambda{}^\mu$ defines $(k^\mu | k^\nu)$, by analogy with $g^{\mu\nu}$, and

$$(k_\nu | k_\lambda) \equiv \tfrac{1}{4}[(k_\nu{}^\ddagger k_\lambda) + (\ )^* + (\ )^\ddagger + (\ )^{\ddagger *}] \tag{3.2}$$

Therefore, we must define $D^\mu \psi_a{}^\nu$ such that $D^\mu(\psi_a{}^\nu k_\nu) = (D^\mu \psi_a{}^\nu) k_\nu + \psi_a{}^\nu (D^\mu k_\nu)$ transforms like P^μ, knowing $D^\mu k_\nu$ from Section 2. (We do not apparently need to raise and lower quaternion indices, except for tensors.) A similar procedure would be required for the covariant derivatives of $\psi_\nu{}^\nu$ and a^ν. Care must be exercised in raising and lowering the non-4-vector indices since only $(b_\mu | b_\nu) = g_{\mu\nu}$ commutes with D^λ. The covariant derivatives of $(k_\mu | k_\nu)$, $(l_\mu | l_\nu)$, and $(\mathrm{a}_\mu | \mathrm{a}_\nu)$ must be calculated from their definitions and $D^\mu k_\nu$, $D^\mu l_\nu$, and $D^\mu \mathrm{a}_\nu$.

To illustrate the occurrence of fields involving the $\{\mathrm{a}_\mu\}$ quaternion elements, consider the following equation (Edmonds, 1974a)

$$\begin{aligned} PW &= mcV, \qquad & V' &\equiv \mathscr{L}^* V \mathscr{L} \\ P^\ddagger V &= mcW, \qquad & W' &\equiv \mathscr{L}^\ddagger W \mathscr{L} \end{aligned} \tag{3.3}$$

Here $W = W^\mu \mathrm{a}_\mu$ and $V = V^\mu b_\mu$. Since $\mathrm{a}_\mu \equiv L^\ddagger e_\mu L$, we have $\mathrm{a}_k{}^\ddagger = L^\ddagger e_k{}^\ddagger L = -\mathrm{a}_k$, $a_0{}^\ddagger = L^\ddagger e_0 L = \mathrm{a}_0$, even in curved space. The Lorentz Principle now requires that $m\mathscr{L} = \mathscr{L} m$, which means that $m = m^\ddagger$, i.e., such particles cannot have quaternion mass, contrary to the Dirac equation. This equation can be easily coupled to the photon A in the usual way, and presumably also to the gravity (curvature) equation. Its Maxwell current is, however, different from the Dirac case. We require $\epsilon\psi^* \begin{pmatrix} b_\mu & 0 \\ 0 & b_\mu{}^\ddagger \end{pmatrix} \psi A^\mu$ for the Lorentz Principle.

We have only considered classical wave equations, but second quantization can be introduced in the quaternion formalism (Edmonds, 1974b). It is most natural to postulate commutation relations directly in terms of the invariant quaternion fields, e.g.

$$\psi_a{}^\dagger \psi_a + \psi_a \psi_a{}^\dagger = \delta(x - x'), \qquad \psi_a \psi_\nu + \psi_\nu \psi_a = 0 \tag{3.4}$$

This gives a very complex structure since the k_μ and l_μ already do not commute, because they are quaternions, and now the fields $\psi_a{}^\mu$ and $\psi_\nu{}^\mu$ are also operators. We must define $(\psi_a{}^\mu k_\mu)^\dagger$ in terms of $\psi_a{}^{\mu\dagger}$ and $k_\mu{}^\dagger$ which requires that we consider whether $\psi_a{}^{\mu} k_\mu = k_\mu \psi_a{}^{\mu}$ is going to be valid in a second quantized formalism. It is simplest to accept this and to assume that $k_\mu{}^\dagger = k_\mu$. However,

$$k_\mu{}^\dagger = (L^\ddagger e_\mu)^\dagger = e_\mu{}^\dagger L^{\ddagger\dagger} \tag{3.5}$$

The unquantized analogue of $q^\dagger$ is q^*, therefore we should expect that $e_\mu{}^\dagger = e_\mu$ is valid. It does not appear that $k_\mu{}^\dagger = k_\mu$ is acceptable since L is an operator describing the curvature field. Similar considerations hold for $b_\mu{}^\dagger$, $l_\mu{}^\dagger$, and $a_\mu{}^\dagger$, which can each be expressed in terms of L and $L^\dagger$. Previously, we considered the commutation relations for the gravity field in terms of b_μ and $b_\mu{}^\dagger$ (Edmonds, 1974b), which are not frame invariant fields like the others. We could instead consider the gravity field in terms of L, but this also is not a frame invariant field ($L' = L\mathscr{L}^{-1}$, see equation (4.7)). If we then postulate a commutation relation such as

$$L^\dagger L - LL^\dagger = \delta(x - x') \tag{3.6}$$

it will likely not be frame invariant. Perhaps something drastic like:

$$b^{\mu\dagger} b_\mu - b_\mu b^{\mu\dagger} = \delta(x - x') \tag{3.7}$$

will be appropriate. Certainly gravity is unique in physics, so it should not be surprising that its second quantized form will be unusual. This problem needs much further attention.

4. *General Coordinate Transformations*

In the above discussion of wave equations in curved space-time, we assumed pseudo-euclidian coordinates ($b_\mu \to e_\mu$, as the curvature goes to zero). We now consider how to introduce a general invertable coordinate transformation from this curved space frame. The following will of course apply to the special case of flat space transformations.

The actual coordinate transformation $x^{\mu'} = x^{\mu'}(\{x^\nu\})$ is not 'of interest' as much as is the transformation relations for such things as A^μ, $\psi_a{}^\nu$, or l_λ. We therefore approach the transformation in an unorthodox manner. We say that a coordinate transformation is 'specified' by a quaternion $\mathscr{L} = \mathscr{L}^\mu(x) e_\mu$, $\mathscr{L}^\mu = \mathscr{L}_R{}^\mu + i\mathscr{L}_I{}^\mu$, i.e. by eight real functions of x. We consider only invertable transformations, meaning that $\mathscr{L}^{-1}$ exists such that $\mathscr{L}\mathscr{L}^{-1} = e_0$.

We then define the new quanternion basis elements as follows:

$$\mathscr{L}^* b_\mu{}' \mathscr{L} \equiv b_\mu, \qquad \mathscr{L}^\ddagger k_\mu{}' \equiv k_\mu$$
$$\mathscr{L}^* l_\mu{}' \equiv l_\mu, \qquad \text{and} \qquad \mathscr{L}^\ddagger a_\mu{}' \mathscr{L} \equiv a_\mu \tag{4.1}$$

These can be inverted using $(\mathscr{L})^{-1}$, $(\mathscr{L}^*)^{-1}$, and $(\mathscr{L}^\ddagger)^{-1}$. Since members of $\{b_\mu{}'\}$ are linear combinations of the members of $\{b_\mu\}$, we can define $a^\nu{}_\mu$ and $a_\nu{}^\mu$ by

$$b_\mu{}' \equiv a_\mu{}^\nu b_\nu \qquad \text{and} \qquad b_\nu \equiv a^\mu{}_\nu b_\mu{}' \tag{4.2}$$

We then find that $a^{\mu}{}_{\nu}a_{\mu}{}^{\lambda} = \delta_{\nu}{}^{\lambda}$ and $a^{\mu}{}_{\nu}a_{\lambda}{}^{\nu} = \delta_{\lambda}{}^{\mu}$. Further, we can determine $\{a^{\mu}{}_{\nu}\}$ from $\{\mathscr{L}^{\nu}, b_{\mu}{}^{(\alpha)}\}$, since

$$\mathscr{L}^* b_{\mu}{}' \mathscr{L} = \mathscr{L}^* a_{\mu}{}^{\nu} b_{\nu} \mathscr{L} = b_{\mu} \Rightarrow \mathscr{L}^* b_{\lambda} \mathscr{L} = a^{\mu}{}_{\lambda} b_{\mu} \Rightarrow \mathscr{L}^{\mu *} \mathscr{L}^{\nu} b_{\lambda}{}^{(\alpha)} e_{\mu} e_{\alpha} e_{\nu}$$
$$= a^{\mu}{}_{\lambda} b_{\mu}{}^{(\alpha)} e_{\alpha} \tag{4.3}$$

We make contact with the actual coordinate transformation by defining:

$$dx^{\mu'} \equiv a^{\mu}{}_{\lambda}\, dx^{\lambda} = \frac{\partial x^{\mu'}}{\partial x^{\lambda}}\, dx^{\lambda} \tag{4.4}$$

This gives a set of coupled partial differential equations for the new variables $x^{\mu'}$ in terms of the old, x^{μ}. The new metric is given by $g_{\mu\nu}{}' \equiv (b_{\mu}{}' | b_{\nu}{}')$.

There must exist a quaternion $L'(x)$ which gives the $b_{\mu}(x)' = b_{\mu}{}^{(\alpha)'} e_{\alpha}$ directly due to the space-time curvature. Therefore, we postulate that:

$$b_{\mu}{}' \equiv L'^* e_{\mu} L' \tag{4.5}$$

and find that

$$b_{\mu} = \mathscr{L}^* b_{\mu}{}' \mathscr{L} = \mathscr{L}^*(L'^* e_{\mu} L') \mathscr{L} = (L'\mathscr{L})^* e_{\mu} (L'\mathscr{L}) \tag{4.6}$$

Therefore, we can calculate L' from

$$b_{\mu} = L^* e_{\mu} L = (L'\mathscr{L})^* e_{\mu} (L'\mathscr{L}) \Rightarrow L' = L\mathscr{L}^{-1} \tag{4.7}$$

For the other quaternion basis elements we find analogous formulas:

$$\mathscr{L}^{\ddagger} k_{\mu}{}' = k_{\mu} = \mathscr{L}^{\ddagger} L'^{\ddagger} e_{\mu} = L e_{\mu} \Rightarrow k_{\mu}{}' = L'^{\ddagger} e_{\mu} \tag{4.8}$$

is consistent with $L' = L\mathscr{L}^{-1}$, which was needed for $b_{\mu}{}'$.

To complete the coordinate transformation discussion, we must specify how the wave function components transform. These are defined such that the quaternion wave functions are frame invariant. The required transformation laws are as follows:

$$A^{\mu'} b_{\mu} \equiv \mathscr{L}^* A^{\mu} b_{\mu} \mathscr{L}, \qquad \psi_a{}^{\mu'} k_{\mu} \equiv \mathscr{L}^{\ddagger} \psi_a{}^{\mu} k_{\mu}$$
$$\psi_v{}^{\mu'} l_{\mu} \equiv \mathscr{L}^* \psi_v{}^{\mu} l_{\mu}, \qquad \text{and} \qquad a^{\mu'} \mathrm{a}_{\mu} \equiv \mathscr{L}^{\ddagger} a^{\mu} \mathrm{a}_{\mu} \mathscr{L} \tag{4.9}$$

These give the new wave function components in terms of the old ones and the coordinate transformation matrix, e.g. $\{\psi_a(x)^{\mu'}\}$ is given in terms of $\{\mathscr{L}(x)^{\mu}, \psi_a(x)^{\mu}\}$. We must now show that this transformation law leaves the quaternions invariant. We have

$$\mathscr{L}^{\ddagger}(\psi_a{}^{\mu'} k_{\mu}{}') = \psi_a{}^{\mu'} \mathscr{L}^{\ddagger} k_{\mu}{}' = \psi_a{}^{\mu'} k_{\mu} = \mathscr{L}^{\ddagger}(\psi_a{}^{\mu} k_{\mu})$$
$$\mathscr{L}^*(\psi_v{}^{\mu'} l_{\mu}{}') = \psi_v{}^{\mu'} \mathscr{L}^* l_{\mu}{}' = \psi_v{}^{\mu'} l_{\mu} = \mathscr{L}^*(\psi_v{}^{\mu} l_{\mu})$$
$$\mathscr{L}^*(A^{\mu'} b_{\mu}{}')\mathscr{L} = A^{\mu'} \mathscr{L}^* b_{\mu}{}' \mathscr{L} = A^{\mu'} b_{\mu} = \mathscr{L}^*(A^{\mu} b_{\mu})\mathscr{L}$$
$$\mathscr{L}^{\ddagger}(a^{\mu'} \mathrm{a}_{\mu}{}')\mathscr{L} = a^{\mu'} \mathscr{L}^{\ddagger} \mathrm{a}_{\mu}{}' \mathscr{L} = a^{\mu'} \mathrm{a}_{\mu} = \mathscr{L}^{\ddagger}(a^{\mu} \mathrm{a}_{\mu})\mathscr{L} \tag{4.10}$$

which ' shows' that the quaternions are invariant as desired. It is also easy to show that these definitions have the required group structure: $x \to x'$ with $\mathscr{L}$ and $x' \to x''$ with $\mathscr{M} \Rightarrow x \to x''$ with $\mathscr{N} = \mathscr{M}\mathscr{L}$. Proof:

$$\psi_a^{\mu''} k_\mu' = \mathscr{M}^\ddagger \psi_a^{\mu'} k_\mu'$$

$$\mathscr{L}^\ddagger \psi_a^{\mu''} k_\mu' = \psi_a^{\mu''} \mathscr{L}^\ddagger k_\mu' = \psi_a^{\mu''} k_\mu = \mathscr{L}^\ddagger \mathscr{M}^\ddagger \psi_a^{\mu'} (\mathscr{L}^\ddagger)^{-1} \mathscr{L}^\ddagger k_\mu'$$

$$= (\mathscr{M}\mathscr{L})^\ddagger (\mathscr{L}^\ddagger)^{-1} \psi_a^{\mu'} k_\mu = (\mathscr{M}\mathscr{L})^\ddagger (\mathscr{L}^\ddagger)^{-1} \mathscr{L}^\ddagger \psi_a^{\mu} k_\mu,$$

$$\Rightarrow \psi_a^{\mu''} k_\mu = (\mathscr{M}\mathscr{L})^\ddagger \psi_a^{\mu} k_\mu \tag{4.11}$$

We now have a way of writing quaternion wave equations in curvilinear coordinates and curved space-time. The problem of second quantization for gravity needs further work but this structure seems to provide a beautiful and natural framework for solving this important problem.

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References

Brill, D. R. and Wheeler, J. A. (1957). *Review of Modern Physics,* **29**, 465.

Edmonds, D. J., Jr. (1972). *International Journal of Theoretical Physics,* Vo. 6, No. 3, p. 205.

Edmonds, J. D., Jr. (1973a). *International Journal of Theoretical Physics,* Vol. 7, No. 6, p. 475.

Edmonds, J. D., Jr. (1973b). *Foundations of Physics,* **3**, 313.

Edmonds, J. D., Jr. (1974a). *American Journal of Physics,* **42**, 220.

Edmonds, J. D., Jr. (1974b). *Foundations of Physics,* to be published.

Rastall, P. (1964). *Reviews of Modern Physics,* **36**, 820.